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Mu-Chun Chen[a] **and Stephen F. King**[b]

[a]*Department of Physics & Astronomy, University of California,*
Irvine, CA 92697-4575, U.S.A.

[b]*School of Physics & Astronomy, University of Southampton,*
Southampton, SO17 1BJ, U.K.

E-mail: muchunc@uci.edu, sfk@phys.soton.ac.uk

ABSTRACT: We introduce the idea of Form Dominance in the (type I) see-saw mechanism, according to which a particular right-handed neutrino mass eigenstate is associated with a particular physical neutrino mass eigenstate, leading to a form diagonalizable effective neutrino mass matrix. Form Dominance, which allows an arbitrary neutrino mass spectrum, may be regarded as a generalization of Constrained Sequential Dominance which only allows strongly hierarchical neutrino masses. We consider alternative implementations of the see-saw mechanism in minimal A_4 see-saw models and show that such models satisfy Form Dominance, leading to neutrino mass sum rules which predict closely spaced neutrino masses with a normal or inverted neutrino mass ordering. To avoid the partial cancellations inherent in such models we propose Natural Form Dominance, in which a different flavon is associated with each physical neutrino mass eigenstate.

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Contents

1 Introduction

The most remarkable discovery in particle physics over the past decade has been the discovery of neutrino mass and mixing involving two large mixing angles commonly known as the atmospheric angle θ_{23} and the solar angle θ_{12}. The latest data from neutrino oscillation experiments is consistent with the so called tri-bimaximal mixing (TBM) mixing pattern [1],

$$U_{\text{TBM}} = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{1.1}$$

The question of how to achieve TBM has been the subject of intense theoretical speculation and there have been many attempts to derive TBM from models based on an underlying family symmetry spontaneously broken by new Higgs fields called "flavons" [2–12]. Since the forthcoming neutrino experiments will be sensitive to small deviations from TBM, it is important to study the theoretical uncertainty in such TBM predictions, and this has also been addressed [13].

Although in the above theoretical models [2–12] the neutrino and charged lepton mass matrices are always constructed in some particular basis, the physical results must always be basis invariant. For example, models of TBM based on the discrete family symmetry group A_4 were originally constructed in a basis in which both the neutrino and charged lepton mass matrices are both non-diagonal [3], but were subsequently reformulated in the more convenient flavour basis in which the charged lepton masses were diagonal [4]. Similarly, when the see-saw mechanism is considered, the Dirac neutrino and right-handed

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neutrino Majorana mass matrices are also constructed in a particular basis but again the results must be basis invariant. Thus, although see-saw models of TBM based on the discrete family symmetry group A_4 have so far been constructed in a basis in which the right-handed neutrino mass matrix is not diagonal [4], in this paper we shall find it convenient to consider such models in the diagonal right-handed neutrino mass basis.

In this paper we introduce the idea of Form Dominance (FD) in the (type I) see-saw mechanism as a generic and natural mechanism which leads to a form diagonalizable effective neutrino mass matrix in which the mixing matrix is independent of the parameters which control the physical neutrino masses. It is well known that models which reproduce a form diagonalizable effective neutrino mass matrix can provide a natural explanation of TBM without any fine-tuning of parameters [14]. Here we shall show how to achieve such a form diagonalizable effective neutrino mass matrix starting from the type I see-saw mechanism using the FD mechanism. The basic idea of FD is that a particular right-handed neutrino mass eigenstate is associated with a particular physical neutrino mass eigenstate, similar to the case of Constrained Sequential Dominance (CSD) [7, 15]. However, whereas CSD only applies to the case of a strong neutrino mass hierarchy, FD is more general and allows three physical neutrino masses with arbitrary masses and ordering. As an example of FD we shall consider minimal A_4 see-saw models, in which the neutrino sector involves only one triplet flavon plus one singlet flavon, including both the usual see-saw model proposed in [4] and a new alternative one. Working in the diagonal right-handed neutrino mass basis, we shall show that both these see-saw models satisfy FD leading to neutrino mass sum rules which predict closely spaced neutrino masses with a normal or inverted neutrino mass ordering [10]. The results motivate the idea of Natural Form Dominance (NFD), in which a different flavon is associated with each physical neutrino mass eigenstate, with CSD as a special case of NFD.

The remainder of the paper is organized as follows. In section 2 we give the form of the TBM effective neutrino mass matrix in the flavour basis where it may be expressed in terms of neutrino masses and columns of the MNS matrix. In section 3 we introduce the idea of FD in a particular basis, then in a basis invariant way. In section 4 we examine the minimal A_4 see-saw models defined above in the diagonal right-handed neutrino mass basis (not usually considered) and show that they satisfy FD, with the neutrino masses obeying various sum rules corresponding to closely spaced neutrino masses with a normal or inverted neutrino mass ordering. Section 5 is reserved for a discussion of our results, including the motivation for NFD, and the conclusion.

2 TBM in the flavour basis

In the flavour basis, in which the charged lepton mass matrix is diagonal and the TBM arises from the neutrino sector, the effective neutrino mass matrix corresponding to TBM, denoted by $(M_{\text{eff}}^\nu)^{\text{TBM}}$, may be diagonalized as,

$$M_{\text{eff}}^{\nu\,\text{diag}} = U_{\text{TBM}}^T (M_{\text{eff}}^\nu)^{\text{TBM}} U_{\text{TBM}} = (m_1,\ m_2,\ m_3)\,. \tag{2.1}$$

Given U_{TBM}, this enables $(M_{\text{eff}}^\nu)^{\text{TBM}}$ to be determined in terms of neutrino masses,

$$(M_{\text{eff}}^\nu)^{\text{TBM}} = m_1 \Phi_1 \Phi_1^T + m_2 \Phi_2 \Phi_2^T + m_3 \Phi_3 \Phi_3^T\,, \tag{2.2}$$

where the three matrices are

$$\Phi_1 \Phi_1^T = \frac{1}{6} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix}, \quad \Phi_2 \Phi_2^T = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \Phi_3 \Phi_3^T = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} , \quad (2.3)$$

corresponding to the orthonormal column vectors

$$\Phi_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \qquad \Phi_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \Phi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \qquad (2.4)$$

Note that $\Phi_{1,2,3}$ are just the three columns of U_{TBM}, namely,

$$\Phi_{1i} = U_{\text{TBM}i1}, \qquad \Phi_{2i} = U_{\text{TBM}i2}, \qquad \Phi_{3i} = U_{\text{TBM}i3}. \qquad (2.5)$$

From above we may write $(M_{\text{eff}}^\nu)^{\text{TBM}}$ as the symmetric matrix,

$$(M_{\text{eff}}^\nu)^{\text{TBM}} = \begin{pmatrix} a & b & c \\ . & d & e \\ . & . & f \end{pmatrix}, \qquad (2.6)$$

where,

$$a = \frac{2}{3}m_1 + \frac{1}{3}m_2,$$
$$b = c = -\frac{1}{3}m_1 + \frac{1}{3}m_2,$$
$$d = f = \frac{1}{6}m_1 + \frac{1}{3}m_2 + \frac{1}{2}m_3,$$
$$e = a + b - d. \qquad (2.7)$$

In particular $b = c$ and $d = f$ and $e = a + b - d$ are the characteristic signatures of the TBM neutrino mass matrix in the flavour basis.

3 Form dominance

The key requirement of a form diagonalizable effective neutrino mass matrix is the presence of no more than three free parameters in the matrix, which are subsequently related to the physical neutrino mass eigenvalues. For example a form diagonalizable effective neutrino mass matrix in the notation of eq. (2.6) involving only three free parameters a, b, d and taking the form

$$(M_{\text{eff}}^\nu)^{\text{TBM}} = \begin{pmatrix} a & b & b \\ . & d & (a+b-d) \\ . & . & d \end{pmatrix}, \qquad (3.1)$$

will result in TBM independently of the parameters a, b, d and hence independently of the physical neutrino masses which are related to the parameters a, b, d by eq. (2.7). On the

other hand if there are more than three free parameters then one or more of the conditions in eq. (2.7) $b = c$ and $d = f$ and $e = a+b-d$ would have to be achieved by tuning and such a matrix would then not be form diagonalizable since the mixing matrix would depend on the parameter choice (and hence depend on the physical neutrino masses). It is clear that the notion of a form diagonalizable effective neutrino mass matrix is related to its dependence on only three (or less) free parameters. If this matrix arises from the type I see-saw mechanism, it is not *a priori* obvious how the underlying theory, involving the Dirac neutrino mass matrix and heavy Majorana neutrino mass matrix could naturally lead to such a form diagonalizable effective neutrino mass matrix. In general, the see-saw mechanism involves many parameters which could enter into the effective neutrino mass matrix arising from the non-symmetric Dirac mass matrix as well as the (typically) three right-handed neutrino masses. To achieve a form diagonalizable effective neutrino mass matrix it is clearly necessary to constrain the form of the Dirac neutrino mass matrix, and also associate the right-handed neutrino masses with the Dirac mass matrix, in such a way that only three (or fewer) independent combinations of parameters enter the effective neutrino mass matrix.

Constrained Sequential Dominance (CSD) [7, 15] provides an example of how this may be achieved for the case of strongly hierarchical neutrino masses. According to CSD, in the diagonal right-handed neutrino mass basis, each column of the Dirac mass matrix is associated with a particular right-handed neutrino mass, and CSD then imposes the constraint that these columns are proportional to those in eq. (2.4), leading to only three independent parameters entering the effective neutrino mass matrix, which is consequently form diagonalizable. However CSD assumes a strong physical neutrino mass hierarchy, $|m_1| \ll |m_2| < |m_3|$ so that effectively the subdominant column associated with m_1 may be neglected, and then only two free parameters associated with two right-handed neutrinos responsible for m_2 and m_3 remain [7]. Here we shall discuss a generalization of CSD applicable to the case of three physical neutrino masses m_1, m_2, m_3 with arbitrary mass values and mass orderings (including the cases of an inverted hierarchy and quasi-degenerate neutrinos as well as hierarchical neutrinos). In other words we shall propose a more general framework which has all the nice properties of CSD, but which allows a non-hierarchical neutrino mass spectrum.

We now introduce the notion of Form Dominance (FD) in the type I see-saw mechanism as an elegant and generic mechanism for achieving a form diagonalizable effective neutrino mass matrix from the type I see-saw mechanism. FD may be defined in the diagonal right-handed neutrino mass basis and diagonal charged lepton mass basis as follows. To set the notation, recall that, in the type I see-saw mechanism, the starting point is a heavy right-handed Majorana neutrino mass matrix M_{RR} and a Dirac neutrino mass matrix (in the left-right convention) M_D, with the light effective left-handed Majorana neutrino mass matrix M_{eff}^{ν} given by the type I see-saw formula [16],

$$M_{\text{eff}}^{\nu} = M_D M_{RR}^{-1} M_D^T. \tag{3.2}$$

In a basis in which M_{RR} is diagonal, we may write,

$$M_{RR} = \text{diag}(M_A, M_B, M_C) \tag{3.3}$$

and M_D may be written in terms of three general column vectors A, B, C,

$$M_D = (A, B, C). \tag{3.4}$$

The see-saw formula then gives,

$$M_{\text{eff}}^{\nu} = \frac{AA^T}{M_A} + \frac{BB^T}{M_B} + \frac{CC^T}{M_C}. \tag{3.5}$$

Using this notation, FD may now be defined as follows. FD is the requirement that each column of the Dirac mass matrix (in the particular basis defined above) is proportional to a different column of the MNS matrix U,

$$A_i = aU_{i1}, \qquad B_i = bU_{i2}, \qquad C_i = cU_{i3}. \tag{3.6}$$

It is then clear that, although there are six parameters in the see-saw theory, only three independent combinations will enter into the effective neutrino mass matrix, after the type I see-saw mechanism. To be precise, the three constants of proportionality a, b, c in eq. (3.6) combine with the three right-handed neutrino masses $M_{A,B,C}$ to yield three independent combinations of parameters appearing in the effective neutrino mass matrix M_{eff}^{ν} given by the see-saw mechanism in eq. (3.5). Moreover, the resulting M_{eff}^{ν} is form diagonalizable, diagonalized by the MNS matrix U, with the physical neutrino masses m_i given by a^2/M_A, b^2/M_B, c^2/M_C. In such a case, each right-handed neutrino mass eigenstate is clearly associated with a particular physical neutrino mass eigenstate of mass m_i. We emphasize that FD applies to any general MNS mixing matrix U, not just TBM.

The notion of FD may now simply be applied to the special case of TBM, in the particular basis defined above, namely the diagonal right-handed neutrino mass basis and diagonal charged lepton mass basis. Applying the FD conditions in eq. (3.6) to the case of $U = U_{\text{TBM}}$, by comparing eq. (3.5) to eqs. (2.2), (2.3), (2.4) it is clear that an effective neutrino mass matrix of the TBM type $(M_{\text{eff}}^{\nu})^{\text{TBM}}$ may be achieved if

$$A = a\Phi_1 = \frac{a}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \qquad B = b\Phi_2 = \frac{b}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad C = c\Phi_3 = \frac{c}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{3.7}$$

Moreover, the constraints in eq. (3.7) lead to a form diagonalizable $(M_{\text{eff}}^{\nu})^{\text{TBM}}$ diagonalized by U_{TBM} (in this basis) with physical neutrino mass eigenvalues given by $m_1 = a^2/M_A$, $m_2 = b^2/M_B$, $m_3 = c^2/M_C$, as indicated previously.

It is interesting to compare FD for TBM defined above to Constrained Sequential Dominance (CSD) defined in [7, 15]. In CSD a strong hierarchy $|m_1| \ll |m_2| < |m_3|$ is assumed which enables m_1 to be effectively ignored (typically this is achieved by taking M_A to be very heavy leading to a very light m_1) then CSD is defined by only assuming the second and third conditions in eq. (3.7) [7]. Thus CSD is seen to be just a special case of FD corresponding to a strong neutrino mass hierarchy. FD on the other hand is more general and allows any choice of neutrino masses including a mild hierarchy, an inverted hierarchy or a quasi-degenerate mass pattern.

Finally note that FD can also be defined in a basis invariant way as follows. In a general basis, one can always write, without loss of generality

$$\left(A_i M_A^{-1/2} \ B_i M_B^{-1/2} \ C_i M_C^{-1/2} \right) = \left(U_{i1} m_1^{1/2} \ U_{i2} m_2^{1/2} \ U_{i3} m_3^{1/2} \right) R^T \qquad (3.8)$$

where R is the general orthogonal matrix introduced by Casas and Ibarra [17]. Then FD corresponds to the case of R equal to the unit matrix (up to permutations which just corresponds to a relabelling of A, B, C). This was in fact observed in [18][1] but only the limit $m_1 = 0$ was considered corresponding to CSD, where it was noted that the R matrix formalism provides a basis invariant formulation of CSD (since R is basis invariant). Here we allow for general m_i with FD defined by R equal to the unit matrix providing a basis invariant definition of FD. However in practice we shall work in the diagonal right-handed neutrino mass basis and diagonal charged lepton mass basis discussed previously.

4 Minimal A_4 models and form dominance

In this section we discuss Form Dominance (FD) in the framework of minimal A_4 models, where minimal means that the neutrino sector only involves one triplet flavon plus one singlet flavon. We shall work in the diagonal charged lepton mass basis (referred to as the flavour basis). To be as general as possible, we discuss these models independently of a particular mechanism for vacuum alignment or of the symmetries required to enforce the operator structure of the models.

4.1 Minimal A_4 models

The group A_4 is a group that describes even permutations of four objects. It has two generators, S and T, and four inequivalent irreducible representations, 1, $1'$, $1''$ and 3. In the diagonal basis for T, the two generators are given in the triplet representation as,

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \qquad (4.1)$$

where $\omega = e^{2\pi i/3}$. The product rules are $3 \times 3 = 1 + 1' + 1'' + 3_S + 3_A$, and $1^a \times 1^b = 1^{(a+b) \bmod 3}$, where $a, \ b = 0, \ 1, \ 2$ for representation 1, $1'$ and $1''$, respectively. The Clebsch-Gordon coefficients of the above product rules can be found in ref. [4].

Without the right-handed neutrinos, the small neutrino masses can be generated by the dimension-5 operator which breaks both the total and individual lepton numbers,

$$y_{ij} \frac{\overline{\ell_i^c} \ell_j H H}{\Lambda_L} , \qquad (4.2)$$

where ℓ_i ($i = 1, 2, 3$) are the lepton doublets, H is the SM Higgs, y_{ij} are the Yukawa couplings and Λ_L is the cutoff scale for the lepton number violation operator. The tri-bimaximal mixing pattern arises if the three lepton doublets transform as a triplet of A_4,

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[1]See eq. 92 of [18] and the subsequent discussion. Note also that to be fully general the column vectors introduced in eq. (2.4) should have been post multiplied by R^T.

and the three right-handed charged leptons are assigned to be singlets under A_4,

$$L = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} \sim 3 , \qquad e_R \sim 1 , \qquad \mu_R \sim 1'' , \qquad \tau_R \sim 1' . \qquad (4.3)$$

The Lagrangian that gives rise to neutrino masses is

$$\mathcal{L}_{LL} = \frac{\overline{L^c} L H H}{\Lambda_L} \left(\frac{\langle \phi_S \rangle}{\Lambda} + \frac{\langle u \rangle}{\Lambda} \right) , \qquad (4.4)$$

where Λ is the cutoff scale of the A_4 symmetry. The triplet flavon field, $\phi_S \sim 3$, and the singlet flavon field, $u \sim 1$, acquire the following vacuum expectation values (VEVs),

$$\frac{\langle \phi_S \rangle}{\Lambda} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \alpha_s , \qquad \frac{\langle u \rangle}{\Lambda} = \alpha_0 . \qquad (4.5)$$

The VEV $\langle \phi_S \rangle$ breaks the A_4 symmetry down to G_S, which is the subgroup of A_4 generated by the group element S. Upon the electroweak symmetry breaking, the following effective neutrino mass matrix is generated,

$$M_{\text{eff}}^\nu = \begin{pmatrix} 2\alpha_s + \alpha_0 & -\alpha_s & -\alpha_s \\ -\alpha_s & 2\alpha_s & -\alpha_s + \alpha_0 \\ -\alpha_s & -\alpha_s + \alpha_0 & 2\alpha_s \end{pmatrix} \frac{v^2}{\Lambda_L} , \qquad (4.6)$$

where v is the SM Higgs VEV. This mass matrix is form-diagonalizable, i.e. it is always diagonalized, independent of the values for the parameters α_s and α_0, by the tri-bimaximal mixing matrix,

$$M_{\text{eff}}^{\nu\,\text{diag}} = U_{\text{TBM}}^T M_{\text{eff}}^\nu U_{\text{TBM}} = \text{diag}(3\alpha_s + \alpha_0, \ \alpha_0, \ 3\alpha_s - \alpha_0) \cdot \frac{v^2}{\Lambda_L} \equiv (m_1, \ m_2, \ m_3) . \quad (4.7)$$

Because the three mass eigenvalues $m_{1,2,3}$ are determined by two parameters, a and b, there is a sum rule among the three light masses [10],

$$m_1 - m_3 = 2m_2 . \qquad (4.8)$$

Given that the solar mass squared difference is positive, this sum rule leads to a prediction for the normal mass hierarchy in the atmospheric neutrino sector. The charged lepton masses are generated due to the following Lagrangian,

$$\mathcal{L}_{\text{lep}} = \frac{1}{\Lambda} \left(y_e (\overline{\ell} \phi')_1 e_R H + y_\mu (\overline{\ell} \phi')_{1'} \mu_R H + y_\tau (\overline{\ell} \phi')_{1''} \tau_R H \right) . \qquad (4.9)$$

Here the triplet flavon field, ϕ_T, acquires a VEV along the following direction,

$$\langle \phi_T \rangle = \begin{pmatrix} v_T \\ 0 \\ 0 \end{pmatrix} , \qquad (4.10)$$

breaking the A_4 symmetry down to G_T, which is the subgroup generated by T. This leads to a diagonal charged lepton mass matrix, i.e. $V_{e,L} = I$, and thus the PMNS matrix is exactly of the tri-bimaximal form, $U_{\text{PMNS}} = V_{e,L} V_\nu^\dagger = U_{\text{TBM}}$.

4.2 The see-saw mechanism and form dominance in minimal A_4 models

The minimal A_4 see-saw realization of the tri-bimaximal mixing pattern has been discussed before [4], however here we examine such models in the diagonal right-handed neutrino mass basis, which has not been discussed before. We discuss two alternative see-saw realizations of the tri-bimaximal mixing pattern in the basis where the charged lepton mass matrix is diagonal $V_{e,L} = I$, the usual one proposed in [4], and an alternative example which we propose. We shall show that in both examples the FD mechanism is present.

4.2.1 The usual see-saw realization

Since the charged lepton mass matrix is diagonal in our realization, it is generated by the same Lagrangian as given above in eq. (4.9). The three right-handed neutrinos transform as a triplet of A_4,

$$N = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} \sim 3 \, , \tag{4.11}$$

and the right-handed neutrino Majorana mass matrix is generated by,

$$M_{RR} = \overline{N^c} N(\langle \phi_S \rangle + \langle u \rangle) = \begin{pmatrix} 2\alpha_s + \alpha_0 & -\alpha_s & -\alpha_s \\ -\alpha_s & 2\alpha_s & -\alpha_s + \alpha_0 \\ -\alpha_s & -\alpha_s + \alpha_0 & 2\alpha_s \end{pmatrix} \Lambda \, . \tag{4.12}$$

The Dirac neutrino mass matrix is generated by the following interaction,

$$M_D = yH\overline{L}N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} yv \, . \tag{4.13}$$

After the see-saw mechanism takes place, the resulting effective neutrino mass matrix is

$$M_\nu^{\text{eff}} = M_D M_{RR}^{-1} M_D^T = U_{\text{TBM}}^T \text{diag}(m_1, m_2, m_3) U_{\text{TBM}}. \tag{4.14}$$

This effective neutrino mass matrix is diagonalized by U_{TBM} with the mass eigenvalues being

$$\text{diag}(m_1, m_2, m_3) = \left(\frac{1}{3\alpha_s + \alpha_0}, \ \frac{1}{\alpha_0}, \ \frac{1}{3\alpha_s - \alpha_0} \right) \frac{y^2 v^2}{\Lambda} \, . \tag{4.15}$$

The sum rule among the three light neutrino masses is given in this see-saw realization by,

$$\frac{1}{m_1} - \frac{1}{m_3} = \frac{2}{m_2} \, , \tag{4.16}$$

which can lead to both normal and inverted hierarchical mass orderings.

In the see-saw realization of the tri-bimaximal mixing described above, the right-handed neutrino Majorana mass matrix M_{RR} is diagonalized by

$$M_{RR}^{\text{diag}} = U_{\text{TBM}}^T M_{RR} U_{\text{TBM}} = \text{diag}(3\alpha_s + \alpha_0, \ \alpha_0, \ 3\alpha_s - \alpha_0)\Lambda \equiv \text{diag}(M_A, M_B, M_C). \tag{4.17}$$

Rotating to the diagonal basis for the right-handed neutrino mass matrix M_{RR}, the Dirac mass matrix is given by,

$$M'_D = M_D U_{\text{TBM}} = yv \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \equiv (A, \, B, \, C) \,, \quad (4.18)$$

where A, B and C are three column vectors of M'_D. Comparing eq. (4.18) and eq. (3.7) we see that Form Dominance is satisfied with the proportionality constants being, $a = b = -c = yv$, so that in this basis the Dirac mass matrix is in fact exactly proportional to the TBM mixing matrix.[2] Form Dominance is thus at work in this model.

The physical light neutrino masses are given by eq. (4.15), with the mass splittings being controlled by the right-handed neutrino masses in eq. (4.17), $M_A = 3\alpha_s + \alpha_0$ $M_B = \alpha_0$ and $M_C = 3\alpha_s - \alpha_0$. Since these masses are controlled by linear combinations of two VEVs α_0 and α_s, some partial cancellations are required to obtain an acceptable neutrino mass pattern and it is impossible to obtain a strong neutrino mass hierarchy in this model.

4.2.2 An alternative see-saw realization

Alternatively, the see-saw mechanism can be implemented in the following way.[3] Instead of the interactions given in eq. (4.12) and (4.13), consider that the Dirac mass term is generated by

$$M_D = H\overline{L}N \left(\frac{\langle \phi_S \rangle}{\Lambda} + \frac{\langle u \rangle}{\Lambda} \right) = \begin{pmatrix} 2\alpha_s + \alpha_0 & -\alpha_s & -\alpha_s \\ -\alpha_s & 2\alpha_s & -\alpha_s + \alpha_0 \\ -\alpha_s & -\alpha_s + \alpha_0 & 2\alpha_s \end{pmatrix} v \,, \quad (4.19)$$

and the Majorana mass matrix is generated by,

$$M_{RR} = M_R \overline{N^c} N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} M_R \,. \quad (4.20)$$

Similar to the previous consideration, the charged lepton mass matrix is generated by the Lagrangian given in eq. (4.9), and thus it is diagonal. In this case, one can easily check that the neutrino mixing matrix is also of the tri-bimaximal form and the three effective mass eigenvalues are,

$$(m_1, \, m_2, \, m_3) = \left((3\alpha_s + \alpha_0)^2, \, \alpha_0^2, \, (3\alpha_s - \alpha_0)^2 \right) \frac{v^2}{M_R} \,, \quad (4.21)$$

leading to a mass sum rule,

$$\left| |\sqrt{m_1}| - |\sqrt{m_3}| \right| = 2|\sqrt{m_2}| \,, \quad \text{for} \quad (3\alpha_s + \alpha_0)(3\alpha_s - \alpha_0) > 0 \quad (4.22)$$

$$\left| |\sqrt{m_1}| + |\sqrt{m_3}| \right| = 2|\sqrt{m_2}| \,, \quad \text{for} \quad (3\alpha_s + \alpha_0)(3\alpha_s - \alpha_0) < 0 \,. \quad (4.23)$$

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[2]This is a special case. In general FD does not require that the Dirac mass matrix be proportional to the MNS matrix, only that the respective columns be proportional.

[3]For implementation in the diagonal basis for S, see [19].

In this see-saw realization, the RH neutrino mass matrix can be diagonalized by the TBM matrix,

$$U_{\text{TBM}}^T M_{RR} U_{\text{TBM}} = \text{diag}(1, 1, -1) M_R \ . \tag{4.24}$$

In this basis, the Dirac neutrino mass matrix becomes,

$$M_D' = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \text{diag}(3\alpha_s + \alpha_0, \ \alpha_0, \ 3\alpha_s - \alpha_0) v \equiv (A, B, C) \ , \tag{4.25}$$

Similar to the previous case, there exists a correspondence between the vectors A, B, C and $\Phi_{1,2,3}$ with proportionality constants being

$$a = 3\alpha_s + \alpha_0, \ b = \alpha_0, \ c = 3\alpha_s - \alpha_0 \ , \tag{4.26}$$

and thus the FD mechanism is at work. In this case the right-handed neutrino masses are degenerate, and the physical light neutrino masses are proportional to a^2, b^2, c^2. Since a, b, c are controlled by linear combinations of two VEVs α_0 and α_s, as before, some partial cancellations are required to obtain an acceptable neutrino mass pattern and it is again impossible to obtain a strong neutrino mass hierarchy in this model.

5 Discussion and conclusion

The current experimental best fit values for the neutrino mixing angles indicate that the neutrino mixing matrix resembles the TBM form. This suggests an underlying (possibly discrete) family symmetry, such as, for example, A_4. In the most attractive such models, the resulting effective neutrino mass matrix is form diagonalizable, that is to say it involves three (or less) parameters and, in the diagonal charged lepton mass basis, is diagonalized by the TBM matrix, independently of the choice of the parameters.

In this paper we have considered in general terms how such a form diagonalizable effective neutrino mass matrix could result from the type I see-saw mechanism. Clearly a necessary condition is that the type I see-saw mechanism must lead to only three (or less) independent parameters appearing in the effective mass matrix. We have proposed FD as an elegant way to achieve a form diagonalizable effective neutrino mass matrix, starting from the type I see-saw mechanism. According to FD a particular right-handed neutrino mass eigenstate is associated with a particular physical neutrino mass eigenstate in such a way that, in the diagonal right-handed neutrino mass basis, the columns of the Dirac mass matrix are proportional to columns of the MNS matrix, as in eq. (3.6) in general, or eq. (3.7) for the TBM case. FD may be regarded as a generalization of CSD, but whereas CSD assumes a strong neutrino mass hierarchy, FD allows three physical neutrino masses with arbitrary masses and ordering.

As an example of these ideas we have considered minimal A_4 models whose neutrino sector involves only one triplet flavon plus one singlet flavon. We have discussed two different minimal A_4 see-saw models, the "usual" one, and a new "alternative" one. Working

in the diagonal charged lepton and right-handed neutrino mass basis, we have shown that both these see-saw models satisfy the conditions of FD, leading to two different neutrino mass sum rules which both predict closely spaced neutrino masses with a normal or inverted neutrino mass ordering. Despite the fact that they satisfy FD, we have seen that the minimal A_4 models require some partial cancellations in order to obtain an acceptable neutrino mass pattern. This is due to the fact that the light physical neutrino mass eigenvalues m_i are each non-trivial functions of the basic parameters of the model, in particular the two flavon VEVs in the neutrino sector of the minimal A_4 models. While this leads to some welcome predictivity of the neutrino masses, it does mean that partial cancellations between the two flavon VEVs are required to achieve the desired hierarchy between the oscillation parameters $\Delta m^2_{\rm atm}$ and $\Delta m^2_{\rm sol}$.

The most natural way to achieve FD without invoking any cancellations would be to have three triplet flavons $\tilde{\Phi}_1$, $\tilde{\Phi}_2$, $\tilde{\Phi}_3$, in the neutrino sector, whose VEVs $\langle\tilde{\Phi}_1\rangle$, $\langle\tilde{\Phi}_2\rangle$, $\langle\tilde{\Phi}_3\rangle$ are proportional to the three columns of the TBM matrix Φ_1, Φ_2, Φ_3, respectively, as in eq. (2.4). These three flavon VEVs would then form the three columns of the Dirac neutrino mass matrix, in the diagonal charged lepton and right-handed neutrino mass basis, as in eq. (3.7). In such a type I see-saw model, which we refer to as Natural Form Dominance (NFD), each light physical neutrino mass m_i would then be controlled by the VEV of a different flavon $\langle\tilde{\Phi}_i\rangle$, allowing an arbitrary neutrino mass spectrum to be achieved without requiring any partial cancellations of parameters. In fact CSD [7] just corresponds to a special case of NFD corresponding to a strong neutrino mass hierarchy $|m_1| \ll |m_2| < |m_3|$, in which the contribution of the flavon $\tilde{\Phi}_1$ is negligible (and hence may be dropped or replaced by any other flavon) while m_2 is controlled by the flavon $\tilde{\Phi}_2$ and m_3 is controlled by the flavon $\tilde{\Phi}_3$. Examples of such CSD models, with $\tilde{\Phi}_1$ replaced by a flavon whose VEV alignment is approximately proportional to $(0,0,1)^T$, have already been proposed based on SU(3), Δ_{27} [6], SO(3) [7, 8] and A_4 [9]. However NFD models with a general (not necessarily hierarchical) neutrino mass spectrum involving all three flavons $\tilde{\Phi}_i$ have yet to be constructed. This would represent an interesting new direction in model building which goes beyond the minimal A_4 see-saw models considered here.

Acknowledgments

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